A quantization of topological $\mathcal{M}$ theory

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Abstract

A conjecture is made as to how to quantize topological $\mathcal{M}$ theory. We study a Hamiltonian decomposition of Hitchin's 7-dimensional action and propose a formulation for it in terms of 13 first class constraints. The theory has 2 degrees of freedom per point, and hence is diffeomorphism invariant, but not strictly speaking topological. The result is argued to be equivalent to Hitchin's formulation. The theory is quantized using loop quantum gravity methods. An orthonormal basis for the diffeomorphism invariant states is given by diffeomorphism classes of networks of two dimensional surfaces in the six dimensional manifold. The hamiltonian constraint is polynomial and can be regulated by methods similar to those used in LQG.

To connect topological $\mathcal{M}$ theory to full $\mathcal{M}$ theory, a reduction from 11 dimensional supergravity to Hitchin's 7 dimensional theory is proposed. One important conclusion is that the complex and symplectic structures represent non-commuting degrees of freedom. This may have implications for attempts to construct phenomenologies on Calabi-Yau compactifications.

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Contents

1 Introduction

Approaches to quantum gravity have so far fallen into two broad classes, according to whether they are background independent or background dependent. So far most work on string and $\mathcal{M}$ theory has been based background dependent methods and ideas. But it has long been acknowledged that this was a temporary expedient and that the ultimate principles of string theory must be formulated in background independent terms. Meanwhile, a great deal of progress has been made on background independent approaches, including loop quantum gravity[1, 2] , causal sets[3] and lorentzian dynamical triangulations[4].

The results of these, especially loop quantum gravity (LQG), have inspired a few attempts to approach string or $\mathcal{M}$ theory from a background independent perspective[5, 6]. These make use of one of the most powerful observations of LQG, which is that theories of gravity are closely related to topological field theories[2]. The precise relation is that gravitational theories are *constrained topological field theories.* This means that their action is a sum of the action for a BF theory, plus quadratic constraints. These are sometimes called theories of forms, because the metric information is coded into the dynamics of forms[7, 8][1]. This is true of general relativity in all dimensions[9], as well as of supergravity in 11 dimensions[5], so it is a fact that must be relevant for how we formulate $\mathcal{M}$ theory.

Recently Dijkgraaf et al[12] proposed a form of *topological $\mathcal{M}$ theory*, which is a seven dimensional theory which is hypothesized to unify two six dimensional theories called

[1] In 4 spacetime dimensions, these turn out to be the self-dual two forms of a metric.

topological string theories[2]. This theory is defined by an action proposed by Hitchin[10, 11], and involves only the dynamics of a three-form in seven dimensions. Dijkgraaf et. al. in fact propose that this theory is related by dimensional reduction to topological field theories relevant for three and four dimensional theories. This makes it natural to suggest that the quantization of Hitchin's theory may be accomplished by using background independent methods which have been successfully applied to topological theories and theories of forms in lower dimensions.

In this paper we make a first attempt at such a background independent quantization of topological $\mathcal{M}$ theory. In the next section we propose a form of the theory as a constrained Hamiltonian theory. We find that the dynamical variables are coded into a two form β and densitized bivector, π, on a compact six manifold Σ. These are canonically conjugate to each other and are associated with the specification of two structures that go into the definition of a Calabi-Yau manifold, which are, respectively, a complex and symplectic structure. We find a system of first class constraints relating them, which we argue is equivalent to the dynamics described earlier by Hitchin in [10, 11].

In section 3 we count the local degrees of freedom, using standard methods. We find there are two local degrees of freedom per point. Thus, if the proposal made in this paper is correct, topological $\mathcal{M}$ theory is not actually a topological field theory.

In section 4 we then quantize the local degrees of freedom, following the methods of LQG. We find a theory of extended objects living in the six dimensional manifold, Σ. These are described by observables parameterized by membranes and four dimensional branes in Σ. These involve, respectively, the complex structure and symplectic structures on Σ. We find that the quantum states of the theory have a separable basis in one-to-one correspondence with the diffeomorphism equivalence classes of the membranes embedded in the six manifold.

In the classical theory of Hitchin, the complex and symplectic structures each give a volume to Σ, and these are required to be equal to each other. In the Hamiltonian formulation presented here, this condition is expressed by a hamiltonian constraint. Its quantization leads to analogues of the Wheeler-deWitt equations. This has a form not seen before, being cubic rather than quadratic in momenta. We are able to use LQG methods to express the WdW operator as a limit of a sequence of regulated operators. Unlike LQG, the operator is the sum of two terms, and no easy solutions are apparent.

Finally, in section 5, we show how the degrees of freedom of Hitchin's theory arise from a dimensional reduction of 11 dimensional supergravity in which the frame fields are set to zero.

While these results may be seen as a first sketch of a quantum theory, there is one intriguing question, raised in [12], that confronts us. The definition of a Calabi-Yau manifold requires fixing both the complex and symplectic structures. Here we find that those structures do not commute with each other quantum mechanically. Thus, the use of Calabi-Yau manifolds to describe compactifications of string and $\mathcal{M}$ theory can only be

[2]Related papers are [13].

sensible at a semiclassical level in which one works on a fixed, classical background geometry. Once quantum gravity effects are turned on, an uncertainty principle may prevent a quantum state as being identified as a Calabi-Yau manifold. This is true in Hitchin's theory, as pointed out in [12], but the fact that the degrees of freedom of that theory arise from a compactification of 11 dimensional supergravity suggest it will be true also in $\mathcal{M}$ theory.

This gives rise to several fascinating questions that future work may address.

- Might there be quantum effects of order l_{Pl} that arise from the quantum fluctuations of the Calabi-Yau geometry? Could this lead to new kinds of effects, perhaps observable in experiments such as AUGER and GLAST?
- Might the quantum fluctuations in the Calabi-Yau geometries help to stabilize then quantum mechanically against decay to the negative energy density states found by [15]?
- If the Calabi-Yau compactifications do not correspond to quantum states of the fundamental theory, but only arise in the classical limit, there are implications for how they are to be counted in considerations of the landscape of theories.

2 Hamiltonian formulation of Hitchin's theory

Hitchin described a seven dimensional theory[10, 11], which Dijkgraaf et al propose is a formulation of topological $\mathcal{M}$ theory[12]. We begin by reviewing their proposal.

2.1 Review of Topological $\mathcal{M}$ theory

The theory is defined on a 7 dimensional manifold, $\mathcal{M}$, There is only one field, which is a real three form Ω, with fixed cohomology class[3].

Analogously to how the metric in LQG in 4d is formed from a set of two forms, we can construct a metric on $\mathcal{M}$, $h(\Omega)$ depending only on Ω. As in the $4d$ case[7, 8], the densitized metric is cubic in the form field. We have

$$\tilde{h}_{ab} = \sqrt{h} h_{ab} = \Omega_{acd}\Omega_{bef}\Omega_{ghi}\epsilon^{cdefghi} \tag{1}$$

The action of Hitchin is

$$I^N = \int_{\mathcal{M}} \sqrt{h(\Omega)} \tag{2}$$

This gives rise to a non-trivial theory when the cohomology class of Ω is frozen. One then has

$$\Omega = \Omega^0 + d\beta \tag{3}$$

[3][12] study the theory in a dual formulation written in terms of a four-form on $\mathcal{M}$.

for β a two form. With Ω^0 fixed the action is a functional of $d\beta$ and hence generates an interesting dynamics.

$$I^N[\beta] = \int_{\mathcal{M}} \sqrt{h(\Omega^0 + d\beta)} \tag{4}$$

Put in this form, the action is invariant under gauge transformations parameterized locally by a one form λ

$$\beta \to \beta' = \beta + d\lambda \tag{5}$$

There are six of these per point of $\mathcal{M}$, , because λ and λ' generate the same gauge transform on β when $\lambda' = \lambda + df$.

2.2 Hamiltonian constrained systems

To quantize any theory we must first cast it into Hamiltonian form[4]. Dirac long ago discovered how to construct a hamiltonian system for a theory invariant under the diffeomorphisms of a $d+1$ dimensional manifold[1]. One considers the manifold to have the form $\Sigma \times R$ where Σ is called the spatial manifold[5]. There are d constraints that generate the diffeomorphisms of Σ, called $\mathcal{D}_i$ where $i, j = 1, ...d$ is a spatial index. There is a Hamiltonian constraint $\mathcal{H}$ that generates the remaining diffeomorphisms in $\Sigma \times R$. Any additional gauge symmetries are generated by constraints $\mathcal{G}$. These constraints must form a first class algebra, which means that they close under Poisson brackets.

In the spatially compact case, which we will assume here, the Hamiltonian must be a linear combination of these constraints. Hence, it must be possible, by a change of variables, to transform the action to the following form,

$$I^H = \int dt \int_\Sigma \left(\pi^{ij} \dot{\beta}_{ij} - \rho_a \mathcal{G}^a - N^i \mathcal{D}_i - N\mathcal{H} \right) \tag{6}$$

where π^{ij} is the momenta conjugate to β_{ij}, while ρ^a, N^i and N are lagrange multipliers.

We next proceed to construct such a theory that we conjecture is equivalent to the theory of Hitchin[10].

2.3 Topological $\mathcal{M}$ theory as a hamiltonian constrained system

The action given by Hitchin can be rewritten in a form suggested by Eli Hawkins[14]. Let g_{ab} be an arbitrary metric on $\mathcal{M}$. Then

$$I^E[g, \Omega] = \int_{\mathcal{M}} \left[\sqrt{g} - g^{ab} \tilde{h}_{ab}(\Omega) \right] \tag{7}$$

[4]Some elements of the hamiltonian formulation were described by Hitchin in [11].

[5]There is no need to assume the degrees of freedom include a metric, so there is not necessarily a distinction between timelike and spacelike.

gives the same equations of motion as I^N when both g and Ω are varied.

Now choose the manifold to be of the form $\mathcal{M} = \Sigma \times R$ with Σ a compact 6 manifold. Form indices in Σ will be denoted, $i, j = 1, ..., 6$. We fix a time coordinate and hence a slicing of $\mathcal{M}$ and define canonical momenta

$$\pi^{ij} = \frac{\delta I^E}{\delta \dot{\beta}_{ij}} \tag{8}$$

where dot denotes as usual derivative by the coordinate on R, called t.

There are 6 initial primary constraints given by

$$\pi^{0i} = \frac{\delta I^E}{\delta \dot{\beta}_{0i}} = 0 \tag{9}$$

The Poisson algebra is generated by

$$\{\beta_{ij}(x), \pi^{kl}(y)\} = \delta^{kl}_{ij} \delta^6(x, y) \tag{10}$$

from which we see that the momenta carry density weight one. This means that the dual is a four-form, $\rho = \pi^*$. In six dimensions, a four form is stable (see [10, 12] for the meaning of this term) and can be written equivalently in terms of a two-form k, as

$$\rho = k \wedge k \tag{11}$$

The theory may then be expressed equivalently in terms of k or π^{ij}. For the canonical quantum theory, the latter is more convenient, as we will see below.

We can see how the action depends on velocities by noting that, in an obvious notation,

$$\begin{aligned} \tilde{h}_{ij} &= (\dot{\beta}_{ij} - d\beta_{oi})(d\beta_{ij})^2 \\ \tilde{h}_{i0} &= (\dot{\beta}_{ij} - d\beta_{oi})^2 d\beta_{ij} \\ \tilde{h}_{00} &= (\dot{\beta}_{ij} - d\beta_{oi})^3 \end{aligned} \tag{12}$$

Hence the action (7) is roughly of the form,

$$I^E \approx \int_\Sigma \int dt \left((\dot{\beta}_{ij} - d\beta_{oi})^3 A_3 + (\dot{\beta}_{ij} - d\beta_{oi})^2 A_2 + (\dot{\beta}_{ij} - d\beta_{oi}) A_1 + \text{potential} \right) \tag{13}$$

where the A_I are polynomials of spatial derivatives of β_{ij}. As a result we will find an equation of the form

$$\pi^{ij} \approx (\dot{\beta}_{kl})^2 A_3 + (\dot{\beta}_{kl}) A_2 + \text{constants} \tag{14}$$

It is not straightforward to invert this relation to find $\dot{\beta}_{kl}$ as a function of π^{ij}. It may be possible to do this, but for the present we proceed by making an educated guess for the form of the Hamiltonian theory based on our experience with other diffeomorphism invariant systems, and checking its internal consistency as well as its agreement with

known results about Hitchin's action. We find such a conjecture, and describe it here. I believe, but have not shown, that the system of constraints described here, is a restatement of previous results on this system[10, 12].

We expect that the inversion of (14) is only possible modulo a system of constraints. This system of constraints will include generators of all *local* gauge invariances of the theory.

We expect a total of 13 first class constraints. Six will generate the gauge transformations (5) in Σ. These must have the form,

$$\mathcal{G}^i = \partial_k \pi^{ik} = 0 \tag{15}$$

These form an abelian algebra.

Six constraints will generate local diffeomorphisms of Σ[6].

They will be given by

$$\mathcal{D}_i = \Omega_{ijk}\pi^{jk} = 0 \tag{16}$$

Let us integrate these against a vector field v^i, with compact support on a contractible region of Σ.

$$\mathcal{D}(v) = \int_\Sigma v^i \Omega_{ijk}\pi^{jk} \tag{17}$$

It is straightforward to express this as

$$\mathcal{D}(v) = \int_\Sigma \left((\mathcal{L}_v \beta_{jk})\pi^{jk} - 2v^i \beta_{ik}\mathcal{G}^k + v^i \Omega^0_{ijk}\pi^{jk} \right) \tag{18}$$

If we ignore the last term, then we see that $\mathcal{D}(v)$ generate a linear combination of diffeomorphisms and gauge transformations (5) on β. However, in a compact, topologically trivial region, we can take $\Omega^0 = d\beta^0$, so that the last term is included in the previous terms. It is then straightforward to show that the algebra of gauge and diffeomorphism constraints (15), and (16), closes, so long as the constraints are multiplied by functions with support on contractible regions.

Now we come to the dynamics. For a diffeomorphism invariant theory on a spatially compact manifold without boundary, this is going to be specified by a hamiltonian constraint $\mathcal{H}$, which must be a local density on Σ. For such a theory it is a general result that the Hamiltonian must be proportional to constraints. The only exception is that there can be a non-vanishing boundary term, but we are considering here the case of a manifold without boundary.

As the action contains terms up to cubic in $\dot{\beta}_{ij}$ we expect $\mathcal{H}$ to have terms up to cubic in π^{ij}. By analogy with the Ashtekar formalism, we may expect that the Hamiltonian constraint will be polynomial in the fields when written as a density of weight two. There

[6] In [12] and [10] another form of the diffeomorphism constraint is proposed. It is plausible, but not yet shown, that the two proposed forms are equivalent, at least on the space of solutions to (15).

are two monomials in π and Ω that give us a scalar of weight two. The first is the simplest scalar density polynomial in the π^{ij}, which is,

$$\mathcal{K} = \pi^{ij}\pi^{kl}\pi^{mn}\epsilon_{ijklmn} \tag{19}$$

This is a kind of kinetic energy. For a potential energy we seek a scalar of density weight two polynomial in the Ω_{ijk}. One exists, defined by Hitchin as follows. Let $\tilde{\kappa}_i^j$ be the densitized, $(1,1)$ tensor,

$$\tilde{\kappa}_i^j = \Omega_{ikl}\Omega_{mno}\epsilon^{klmnoj} \tag{20}$$

Note that the trace $\tilde{\kappa}_i^i = 0$. However the trace of the square is not zero, and it gives a scalar density of weight two,

$$\mathcal{V} = \tilde{\kappa}_i^j\tilde{\kappa}_j^i \tag{21}$$

Combining them, we have a natural candidate for the hamiltonian constraint[7], which is

$$\mathcal{H} = \mathcal{K} - a\mathcal{V} \tag{22}$$

where a is a dimensionless factor.

We can check this guess by seeing if it leads to a constraint algebra that closes. The fact that $\mathcal{H}$ is a scalar density of weight two determines that its Poisson brackets with (15) and (16) closes, so long as the gauge transformations and diffeomorphisms have compact support on contractible regions. To compute the rest of the Poisson algebra we smear against a test function N of density weight minus one, again with compact support in a topologically trivial region.

$$\mathcal{H}(N) = \int_\Sigma N\left(\pi^{ij}\pi^{kl}\pi^{mn}\epsilon_{ijklmn} - a\tilde{\kappa}_i^j\tilde{\kappa}_j^i\right) \tag{23}$$

It is straightforward to check that the algebra closes

$$\{\mathcal{H}(N), \mathcal{H}(M)\} = \int_\Sigma w^j_{NM}\mathcal{D}_j = \mathcal{D}(w_{NM}) \tag{24}$$

where

$$w^j_{NM} = 18a(N\partial_i M - M\partial_i N)\pi^{ik}\tilde{\kappa}_k^j \tag{25}$$

Thus, we see that the combination of the 13 constraints, $\mathcal{G}^i$, $\mathcal{D}_i$ and $\mathcal{H}$ make a closed system of first class constraints.

In fact, we can argue that its solutions are identical to the solutions of Hitchin's theory. When $\mathcal{H} = 0$ we have locally

$$a\tilde{\kappa}_i^j\tilde{\kappa}_j^i = \pi^{ij}\pi^{kl}\pi^{mn}\epsilon_{ijklmn} \tag{26}$$

[7]This is related to a form of the hamiltonian studied by Hitchin in [11].

We can take the square root of each side to find that

$$\sqrt{|a\tilde{\kappa}_i^j\tilde{\kappa}_j^i|} = \sqrt{|\pi^{ij}\pi^{kl}\pi^{mn}\epsilon_{ijklmn}|} \tag{27}$$

We can find a geometric interpretation of the hamiltonian constraint. To do so we note that the Ω is known to characterize the complex structure of Σ [10, 12]. The densitized bivector π^{ij} provides a symplectic structure. These fields allow us to form two different volume elements on the six manifold Σ.

There is a volume element associated with the symplectic structure,

$$\epsilon_\pi = \sqrt{|\pi^{ij}\pi^{kl}\pi^{mn}\epsilon_{ijklmn}|} \tag{28}$$

There is similarly a volume element associated with the complex structure, given by the three form metric $h(\Omega)$, pulled back into the six manifold Σ.

$$\epsilon_h = \sqrt{|\tilde{\kappa}_i^j\tilde{\kappa}_j^i|} \tag{29}$$

The Hamiltonian constraint says that the two volume forms are equal to each other, up to the constant a. In [10, 12] we see that Hitchin's theory implies that

$$2\int_\Sigma \epsilon_h = \int_\Sigma \epsilon_\pi \tag{30}$$

We see that this condition is implied by the guess for the Hamiltonian constraint we gave, (22) so long as $a = \frac{1}{4}$. Hence, the diffeomorphism classes of solutions to the theory given here will coincide with the solutions of Hitchin's theory.

Hitchin[10] also provides a translation to the complex geometry of 6 manifolds. He shows (Proposition 2) that when

$$\mathcal{V} < 0 \tag{31}$$

one can define a complex structure

$$J_i^j = \frac{\tilde{\kappa}_i^j}{\sqrt{-\mathcal{V}}} \tag{32}$$

such that Ω is the real part of a complex holomorphic three-form.

To summarize we have argued that the Hitchin action can be rewritten as a constrained hamiltonian system of form, (6) with constraints given by (15), (16) and (22). We also reach the important conclusion mentioned in the introduction, that the complex and symplectic structures are coded by canonically conjugate degrees of freedom, so long as (31) is imposed.

3 Counting of degrees of freedom

It is straightforward to count the local degrees of freedom. There are 15 β_{ij} which have 15 conjugate momenta π^{ij}. We have 13 first class constraints, which will require 13 gauge fixing conditions. This leaves $2+2$ canonical degrees of freedom. Thus the theory is not topological, there are two local degrees of freedom per point of Σ.

There are, of course, also global degrees of freedom, that correspond to integration of Ω around non-contractible cycles of Σ.

4 Quantization

Dirac proposed a method to quantize hamiltonian constrained systems. With some refinements to take into account issues of regularization and ordering that arise in field theories, this is the method that all background independent approaches to hamiltonian quantization follow. Dirac's method can be further specialized to the case of diffeomorphism invariant theories whose configuration variables are connections or p-forms with local gauge invariance[1, 2]. This specification of Dirac's method to theories invariant under both diffeomorphisms and local gauge invariances is the essence of the Hamiltonian part of loop quantum gravity. We first briefly summarize the procedure, then we apply it to the constrained system just introduced.

4.1 Brief review of Dirac quantization

We begin by specifying the *kinematical configuration space*, $\mathcal{C}$. In the case of topological $\mathcal{M}$ theory, this is the space of two forms β on Σ. By imposing invariance under the action of the gauge and spatial diffeomorphism constraints, in this case (15) and (16), we then go down to a gauge and (spatially) diffeomorphism invariant configuration space

$$\mathcal{C}^{diffeo} = \frac{\mathcal{C}}{\text{local gauge transformations} \times \; Diff(\Sigma)} \tag{33}$$

The aim of the quantization procedure is to first, construct the corresponding Hilbert spaces and, second, construct the Hamiltonian constraint as an operator on diffeomorphism invariant states.

This is accomplished in three steps:

STEP 1: Find an algebra $\mathcal{A}$ of observables on the kinematical phase space which has a representation $\hat{\mathcal{A}}$ on a Hilbert space $H^{kinematical}$ such that

1. The reality conditions of the classical theory, i.e. which variables are real, are realized by the inner product on $H^{kinematical}$. That is, the inner product is chosen so that real classical observables are represented by Hermitian operators.

2. $H^{kinematical}$ carries an exact, non-anomalous unitary representation of $Diff(\Sigma)$. This is given by unitary operators, $\hat{U}(\phi)$, where $\phi \in Diff(\Sigma)$.

STEP 2 Construct a space of diffeomorphism invariant states $H^{diffeo} \subset H^{*}_{kinematical}$, which are invariant under the action of $\hat{U}(\phi)$. These are the diffeomorphism invariant states and they live inside the dual of the kinematical Hilbert space.

STEP 3 Construct a sequence of regularized operators, $\mathcal{H}^{\epsilon}(x)$ to represent the Hamiltonian constraints, in $H^{kinematical}$. Prove that the limit as $\epsilon \to 0$ takes diffeomorphism invariant states to diffeomorphism invariant states, and thus defines a finite operator in H^{diffeo}. Prove that the limit has a kernel in H^{diffeo} that is infinite dimensional. This kernel $H^{physical} \subset H^{diffeo}$ is the physical Hilbert space.

When carried out in LQG there are four key observations, that may extend to the present case

- There is no known way to realize the second condition of **STEP 1** when $\mathcal{A}$ is the usual local canonical algebra defined by the gauge connection and conjugate electric fields. In particular, Fock representations fail because they depend on a background metric, which breaks diffeomorphism invariance. To proceed one must base $\mathcal{A}$ on extended observables, such as Wilson loops.

- When $\mathcal{A}$ is taken to include the Wilson loops of the connection, together with conjugate operators linear in the momenta of the connection, there is a theorem [17] that says that there is a unique way to realize the first two steps. It is not known if this extends to the present case, but if it does there would appear to be only one way to successfully carry out this program for topological $\mathcal{M}$ theory.

- The kinematical Hilbert space $H^{kinematical}$ is not separable, because any two distinct, non-overlapping, Wilson loop operators create orthogonal states. However, this non-separability is exactly cancelled by imposing diffeomorphism invariance. Hence H^{diffeo} is separable, assuming only a technical condition, which is that it is defined in terms of piecewise smooth diffeomorphisms.

- When applied to general relativity, all three steps have been carried out rigorously[16]. ∎

4.2 Quantum topological $\mathcal{M}$ theory

We here sketch how the program just described may be applied to topological $\mathcal{M}$ theory. We do not attempt to give a rigorous treatment, but we find that at a particle physics level of rigor we can follow the same program as was originally used in constructing LQG.

We begin by finding the algebra of observables analogous to Wilson loops and their conjugate variables, to represent the local degrees of freedom. We start with the analogue

of Wilson loops. Given any closed and contractible two surface S in Σ we define a function of $\mathcal{C}$,

$$T[S] = e^{\int_S \beta} \tag{34}$$

Similarly, given a four dimensional surface $A \in \Sigma$ we define momentum flux operators

$$\Pi[A] = \int_A \pi^* \tag{35}$$

where π^* is a four formequivalent to the momenta π. They have a simple Poisson algebra

$$\{T[S], \Pi[A]\} = Int[S, A]T[S] \tag{36}$$

where $Int[S, A]$ is the intersection number of the surfaces S and A.

We note that these observables commute with the action of local gauge transformations generated by (15).

Following the strategy just outlined, we seek a representation of (36) on a Hilbert space, $H^{kinematical}$ that carries a nonanamolous representation of $Diff(\Sigma)$.

Let Γ be a network of two surfaces $S \in \Sigma$, whose faces are labeled by integers. The integers count elementary closed surfaces, out of which the network is formed. This implies that the triangle inequalities are satisfied at every trivalent edge where surfaces meet.

States are functionals of Γ, so we have

$$< \Gamma|\Psi >= \Psi(\Gamma) \tag{37}$$

The operator representing $T[S]$ is defined by

$$< \Gamma| \circ \hat{T}[S] =< \Gamma \oplus S| \tag{38}$$

where $\Gamma \oplus S$ is the network Γ with the surface S added. This gives us

$$\hat{T}[S] \circ \Psi[\Gamma] = \Psi[\Gamma \oplus S] \tag{39}$$

The conjugate momentum operator $\Pi[A]$ is defined by

$$< \Gamma|\hat{\Pi}[A] = \imath\hbar \sum_{S \in \Gamma} Int[S, A] < \Gamma| \tag{40}$$

One can check explicitly that the commutator

$$[\hat{T}[S], \hat{\Pi}[A]] = \imath\hbar Int[S, A]\hat{T}[S]. \tag{41}$$

The kinematical inner product is

$$< \Gamma|\Gamma' >= \delta_{\Gamma\Gamma'} \tag{42}$$

This gives rise to a non-separable Hilbert space, as in LQG. The unitary representation of $\phi \in Diff(\Sigma)$ is defined by

$$< \Gamma|U(\phi) =< \phi \circ \Gamma| \tag{43}$$

This is easily shown to be unitary under (42).

We then define the diffeomorphism invariant Hilbert space, H^{diffeo} to be those states in the dual of H^{kin} such that

$$< \Psi|\hat{U}(\phi) =< \Psi| \tag{44}$$

Following the standard method of LQG, these can be shown to have a countable, orthonormal basis, given by $|\{\Gamma\} >$, where $\{\Gamma\}$ are diffeomorphism classes[8] of networks of labeled two-surfaces embedded in Σ

We now want to introduce a regularized Hamiltonian constraint operator, $\hat{\mathcal{H}}_\epsilon$ expressed in terms of elements of the surface algebra, in the kinematical Hilbert space. This should have several properties:

1. On the classical counterpart, $\lim_{\epsilon \to 0} \mathcal{H}_\epsilon = \mathcal{H}$.

2. The limit $\lim_{\epsilon \to 0} \hat{\mathcal{H}}_\epsilon$ acts on H^{diffeo} in that it takes diffeomorphism invariant states to diffeomorphism invariant states.

Here are some steps towards the construction of such a regularized operator. A regularization procedure is going to break diffeomorphism invariance in Σ. So let us introduce in a local region $\mathcal{R}$, a flat metric q^0_{ij} in Σ and a set of coordinates y^i. At a point $p \in \Sigma$ we can have a box $\mathcal{B}^\epsilon_{\hat{i}\hat{j}\hat{k}}(p)$ of Volume ϵ^3 in q^0_{ij} alongside the $\hat{i}, \hat{j}\hat{k}$ axis. We have, to leading order

$$T[\mathcal{B}^\epsilon_{\hat{i}\hat{j}\hat{k}}(p)] = 1 + \epsilon^3 F_{\hat{i}\hat{j}\hat{k}}(p) \tag{45}$$

where $T[\mathcal{B}^\epsilon_{\hat{i}\hat{j}\hat{k}}(p)]$ takes the intergral of β around the surface of the box.

We can then write a regularized three form operator as

$$\Omega^\epsilon_{\hat{i}\hat{j}\hat{k}} = \frac{1}{\epsilon^3}\left(T[\mathcal{B}^\epsilon_{\hat{i}\hat{j}\hat{k}}(p)] - 1\right) \tag{46}$$

We can then write a regulated Hamiltonian constraint operator

$$\mathcal{H}^\epsilon = \mathcal{K}^\epsilon + \mathcal{V}^\epsilon \tag{47}$$

with

$$\mathcal{V}^\epsilon = \tilde{\kappa}^{\epsilon j}_i \tilde{\kappa}^{i\epsilon}_j \tag{48}$$

[8]To eliminate continuous labels on states coming from labeling diffeomorphism equivalence classes of complicated intersections, the diffeomorphisms are extended to piecewise smooth diffeomorphisms, after which the basis is countable[1].

where the regulated operator $\tilde{\kappa}_j^{i\epsilon}$ is

$$\tilde{\kappa}_{\hat{i}}^{\hat{j}\epsilon} = \Omega^{\epsilon}_{\hat{i}\hat{k}\hat{l}}\Omega^{\epsilon}_{\hat{m}\hat{n}\hat{o}}\epsilon^{\hat{k}\hat{l}\hat{m}\hat{n}\hat{o}\hat{j}} \tag{49}$$

Similarly, let us define a surface $A_{\epsilon}^{\hat{i}\hat{j}}(p)$ to be a four dimensional hypercube of size ϵ on a side, orthogonal to the $\hat{i}\hat{j}$ directions, all with respect to the background metric q^0_{ij}, at the point p. We then can define

$$\hat{\Pi}_{\epsilon}^{\hat{i}\hat{j}} = \frac{1}{\epsilon^4}\Pi(A_{\epsilon}^{\hat{i}\hat{j}}(p)) \tag{50}$$

We then have for the regularized kinetic energy

$$\mathcal{K}^{\epsilon} = \epsilon_{\hat{i}\hat{j}\hat{k}\hat{l}\hat{m}\hat{n}}\hat{\Pi}_{\epsilon}^{\hat{i}\hat{j}}\hat{\Pi}_{\epsilon}^{\hat{k}\hat{l}}\hat{\Pi}_{\epsilon}^{\hat{m}\hat{n}} \tag{51}$$

There remains much to do, but the outline is clear from here, by analogy with the development of LQG. For example, one can define a path integral by exponentiation. It will be defined as a spin foam model, based on labeled triangulations of $\mathcal{M}$. The three-simplices of the triangulation will be labeled with integers, corresponding to the evolutions of the graphs. There will also be labels on the four-simplices, corresponding to π^*.

5 Down from 11 dimensions

We do not have a background independent formulation of $\mathcal{M}$ theory, so the existence of the theory remains a conjecture. But part of that conjecture is that a classical limit of $\mathcal{M}$ theory is given by 11 dimensional supergravity. Hence it is of interest to see if Hitchin's 7 dimensional theory might be derived from a suitable reduction of 11 dimensional supergravity. If it can be, then it may be that we can identify the quantum states just described as the actual quantum degrees of freedom corresponding to the membranes of $\mathcal{M}$ theory.

As is the case in all known versions of general relativity and supergravity, the action and field equations for 11 dimensional supergravity can be written in a polynomial form [5]. This makes it possible to take a consistent reduction in which the frame field, connection and gravitino fields (with certain density weights) are taken to zero, leaving only the three form a_{ABC}[9] Since the field equations are polynomial, these provide a subset of solutions to the full equations of $11d$ supergravity. It can be shown that the supersymmetry transformations, which are also polynomial, are trivially satisfied for such solutions.

In this reduction, the action is

$$I^{11} = \int_{\mathcal{M}^{11}} da \wedge da \wedge a \tag{52}$$

This is a version of higher dimensional Chern-Simons theory. Its dynamics and quantization were studied in detail in [5]. It is important to note that higher dimension and

[9] $A, B, ... = 0, 1, ..., 10$, while ten dimensional spatial indices are given by $I, J, ... = 1, ..., 10$.

higher form Chern-Simons theories have local degrees of freedom. This theory is diffeomorphism invariant, but it is not topological.

Let us see if the degrees of freedom of Hitchin's 7 dimensional theory can be found imbedded in this metric-less reduction of 11 dimensional supergravity.

The field equations of (52) are

$$da \wedge da = 0 \tag{53}$$

The gauge transformation is of course

$$\delta a = d\lambda \tag{54}$$

with λ a two form.

A solution to (53) is given by the following ansatz. Let $A = a, \alpha$, with $a = 0, ..., d$ and $\alpha = d+1, ...10$, Then the ansatz is

$$da_{\alpha ABC} = 0 \tag{55}$$

so long as $d \leq 6$. It is interesting that the largest non-trivial case is $d = 6$, which gives us a reduction to a seven dimensional theory.

In fact, we can find a simple set of solutions, that are locally but not globally flat. Let the topology be chosen to be the standard one proposed for a reduction from $\mathcal{M}$ theory to string theory,

$$\mathcal{M}^{11} = R \times \Sigma \times S^1 \times R^3. \tag{56}$$

Here Σ is a compact six manifold, and the R is time, as in previous sections. These are coordinatized as before by x^a, $a = 0, i$, with $i = 1, ..., 6$. Let $y^7 = \theta$ be the coordinate around the S^1. This is the standard circle around which membranes are wrapped to get strings. The three remaining dimensions in the R^3 can as usual be taken to be ordinary, uncompactified space, coordinatized by y^α with $\alpha = 1, 2, 3$ from now on. We will assume that everything is constant in space, so that

$$\frac{\partial a_{ABC}}{\partial y^\alpha} = 0 \tag{57}$$

This, physically, means that we are studying the geometry of string compactifications that might arise from $\mathcal{M}$ theory.

Let us take a solution which is locally pure gauge, of the form of (54), with (locally on the S^1)

$$\lambda_{ab} = \theta \gamma_{ab}(x) \tag{58}$$

However globally, we will have

$$\beta_{ab}(x) = \oint_{S^1} d\theta a_{\theta ab} = \gamma_{ab}(x) \tag{59}$$

Since the solution is locally trivial, $da = 0$ everywhere, so this is a solution to $11d$ supergravity.

From (54) we see that there is still a gauge invariance, given by

$$\delta\beta = \delta\gamma = d\phi \tag{60}$$

where ϕ is a one form.

The integral around three-cycles C_I of Σ are given by

$$\int_{C_I} a = \int_{C_I} d\beta \tag{61}$$

These are constants, as they do not evolve in time under the equations of motion (53).

The canonical momenta for a_{IJK} is $\Pi^{IJK} \sim (a \wedge da)^*$, where the duality is in the ten dimensional manifold $\Sigma \times S^1 \times R^3$. We have

$$\{a_{IJK}(x), \Pi^{LMN}(y)\} = \delta^{10}(x,y)\delta^{LMN}_{IJK} \tag{62}$$

The dimensionally reduced momenta is

$$\pi^* = \int_{R^3} \Pi^* \tag{63}$$

which is a four-form on $\Sigma \times R$. It can be pulled back to a four form on Σ. We have

$$\{\beta_{ij}(x^i), \pi^*_{klmn}(y^i)\} = \int_{S^1} d\theta \int_{R^3} d^3x^{\alpha\beta\gamma}\{a_{\theta ij}, \Pi^*_{klmn\alpha\beta\gamma}\} = \delta^6(x^i, y^i)\epsilon_{ijklmn} \tag{64}$$

Thus, the canonical degrees of freedom of topological $\mathcal{M}$ theory can be seen to arise from the reduction of supergravity from 11 dimensions.

The reduced theory then has degrees of freedom (β_{ab}, π^{cd}), with fixed cohomology on $R \times \Sigma$. In the quantum theory there will arise an effective action to describe the low energy dynamics of these degrees of freedom. The effective action will be dominated by the lowest dimension term that can be made from $d\beta$ on $\Sigma \times R$. One can conjecture that this will be given by the Hitchin's action, which is a cosmological constant term, and hence should dominate the low energy limit.

It is possible we can proceed further in this direction. Let g be a flat metric on the R^4 parameterized by x^0 and y^α and let e^0, e^α be four one form orthonormal frame fields. Given the imbedding of R^4 into $\mathcal{M}^{11}$ we can pull these back to a degenerate set of 11 dimensional frame fields. We may conjecture that these, together with any a_{ABC} such that locally $da = 0$, give solutions of 11 dimensional supergravity. If this is true, then there is a sector of $\mathcal{M}$ theory with a conventional geometry on the four uncompactified spacetime dimensions, but where the geometry on the compactified dimensions is entirely based on a forms theory.

There should be much more in this sector. We should be able to add other degrees of freedom coming from the fields of 11 dimensional supergravity to systematically expand Hitchin's theory to a reduction of $\mathcal{M}$ theory, with a full set of local degrees of freedom.

6 Conclusions

What is described here is a first step towards a background independent quantization of topological $\mathcal{M}$ theory. Many issues remain open. While the conjecture that the constrained system here is equivalent to Hitchin's seven dimensional theory is plausible, it still needs to be proved. The results on the quantum theory are just a first sketch, along the lines of early papers on LQG. It is likely that the quantization can be made rigorous, along the lines of [16]. Of great interest is whether there is an extension of the LOST uniqueness theorem[17] to this context. Further exploration of this direction can be expected to shed light both on the key question of what $\mathcal{M}$ theory may be as well as on the interpretation of the results in LQG concerning $3+1$ dimensional physics. The idea proposed in [12] that Hitchin's theory may open the way to a unification of string theory and LQG is intriguing and the results obtained here give us a common language within which the precise relationship between the two approaches can be elucidated.

But we can already draw a few interesting conclusions from the results obtained here. First we see a possible non-perturbative origin for D-brane states in a background independent formulation. Second, as pointed out in [12], there are implications for string compactifications. In the standard string compactifications on Calabi-Yau manifolds, the Ω and π^{ij} are fixed. However, we see that in topological $\mathcal{M}$ theory these are conjugate variables. Moreover, we see that these variables can be understood to descend from full $\mathcal{M}$ theory, where they are still conjugate variables. If so, then there can be no quantum states of $\mathcal{M}$ theory corresponding to fixed Calabi Yau geometries on Σ. Thus, any phenomenology that depends on the fixed background structure of a Calabi-Yau manifold can only be meaningful in the semiclassical limit in which conjugate variables can both have definite values.

Finally, it is interesting to note that the real variables on which Hitchin's theory is based only define a complex manifold when the condition (31) is satisfied. This is analogous to the condition that the determinant of the spatial metric be positive. It means that the part of the configuration space that corresponds to complex geometries is not a vector space, but satisfies a non-linear inequality. There is then the issue of how this inequality is to be satisfied in the quantum theory. Just as the metric may have an amplitude to be non-degenerate in any first order formulation of quantum gravity, so we must consider the possibility that a quantum state can give a non-zero amplitude to a region of configuration space where (31) is violated, leading to quantum fluctuations in which Σ fails to have a complex structure.

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